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Quantitative Results on Almost Convergence of a Sequence of Positive Linear Operators

R. N. MOHAPATRA

American University of Beirut, Beirut, Lebanon

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1. INTRODUCTION

Let $C(S)$ be the linear space of continuous real-valued functions on a set S, and let $\{L_n\}_{n=1}^{\infty}$ be a sequence of positive linear operators from $C(s)$ into $C(S)$.

After Lorentz [8], we define $L_n(f)$ to be *almost convergent* to g in $C(S)$, *uniformly* in S, provided

$$t_p^k(f)(x) = (1/p) \sum_{n=k+1}^{k+p} L_n(f)(x), \qquad p = 1, 2,..., \quad k = 1, 2,.... \tag{1.1}$$

converges to $g(x)$ when $p \to \infty$, uniformly in k, and uniformly in S, $L_n(f)(x)$ being the value of $L_n(f)$ at the point $x \in S$.

A system $\{f_i\}_{i=0}^{m}$ of functions of $C(S)$ is said to be a set of *test functions* for almost convergence if and only if the almost convergence of $L_n(f_i)$ to f_i $(i = 0, 1,..., m)$ uniformly in S implies the almost convergence of $L_n(f)$ to f, uniformly in S, for all $f \in C(S)$.

With regards to convergence of a sequence of positive linear operators Korovkin [6] showed that for the cases $C(S) = C[a, b]$ and $C(S) = C_{2\pi}$, the linear space of continuous real-valued periodic functions of period 2π, systems of test functions are respectively, $\{1, x, x^2\}$ and $\{1, \sin x, \cos x\}$. It was also shown by him that for $C(S) = C[a, b]$, a necessary and sufficient condition for $\{f_i\}_{i=0}^{2}$ to be a system of test functions is that $\{f_0, f_1, f_2\}$ form a Chebyshev system.

King and Swetits [7] have shown that the results of Korovkin with regard to test functions hold when convergence is replaced by almost convergence.

In the present paper we shall estimate the degree of almost convergence of $L_n(f)$ to f in terms of corresponding test functions. In the beginning of each subsequent section we shall indicate the background against which the problems suggested themselves.

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Since the measure of the degree of almost convergence of $L_n(f)$ to f has to be made in terms of a suitable norm, we define the norm below.

To each $f \in C(S)$ we associate a double sequence $\{t_p^k(f)\}$ as in (1.1) through $L_n(f)$. Writing $t_p(f) = \{t_p^k(f)\}$ for each p, the function $t_p(f) \to \sup_k \sup_{x \in S} |t_p^k(f)(x)|$ defines a norm in the sequence space generated by associating with each $L_n(f)$ the sequence $\{t_p(f)\}$ in the prescribed manner. Let us write $\|t_p(f)\| = \sup_k \sup_{x \in S} |t_p^k(f)(x)|$; $\{L_n(f)\}$ is almost convergent to $g \in C(S)$, uniformly in S, if and only if $\sup_k \sup_{x \in S} |t_p^k(f)(x) - g(x)| \to 0$ as $p \to \infty$ which amounts to the fact that $\|t_p(f) - g\|$ tends to zero as p tends to infinity.

If $\{L_n\}$ is a sequence of positive linear operators then the following hold.

(i) $f \leqslant g$ implies $L_n(f) \leqslant L_n(g)$ for all $f, g \in C(S)$.

(ii) $|f| \leqslant g$ implies $|L_n(f)| \leqslant L_n(|f|) \leqslant L_n(g)$.

(iii) $|f| \leqslant g$ implies $|t_p^k(f)(x)| \leqslant t_p^k(g)(x)$ for all $x \in S$.

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Let $C[a, b]$ be the Banach space of continuous real-valued functions on $[a, b]$ normed by the supremum norm. Let $e^i \in C[a, b]$ be defined by $e^i(x) = x^i$, $i = 0, 1, 2$, for all x in $[a, b]$. $\{L_n\}$ is a sequence of positive linear operators on $C[a, b]$. We would like to estimate $\|t_p(f) - f\|$ and put Theorem 3 of King and Swetits [7] in a quantitative form. Our Theorem 1 can be considered to be an analog of Theorem 1 of Shisha and Mond [10] for almost convergence.

THEOREM 1. *Let $\{L_n\}$ be a sequence of positive linear operators on $C[a, b]$ and let $f \in C[a, b]$ have modulus of continuity w. Let $t_p^k(e^0)(x)$ be uniformly bounded on $[a, b]$ for each p, uniformly in k. Then for $p = 1, 2, \ldots$,*

$$\|f - t_p(f)\| \leqslant \|f\| \, \|t_p(e^0) - 1\| + w(\mu_p) \|t_p(e^0) + 1\|, \tag{2.1}$$

where

$$\mu_p^2 = \|t_p((t - x)^2)(x)\|$$

and

$$\|f\| = \sup_{x \in [a, b]} |f(x)|.$$

If, in particular, $t_p^k(e^0)(x) = 1$, then (2.1) reduces to

$$\|f - t_p(f)\| \leqslant 2w(\mu_p). \tag{2.2}$$

Proof. Proceeding as in Shisha and Mond [10], we have for all $t, x \in [a, b]$ and any positive number δ

$$|f(t) - f(x)| \leqslant (1 + (|t - x|^2/\delta^2))\, w(\delta). \tag{2.3}$$

Hence using the fact that $\{L_n\}$ is a sequence of positive linear operators, we have

$$|t_p^k(f)(x) - f(x)\, t_p^k(e^0)(x)| \leqslant w(\delta)[t_p^k(e^0)(x) + (\mu_p/\delta)^2].$$

If $\mu_p > 0$, choose $\delta = \mu_p$ and (2.1) is now easily seen. If $\mu_p = 0$, use the fact that $w(\delta) \to 0$ as $\delta \to 0^+$ and (2.1) can be proved in this case too.

Remark. We have

$$\mu_p^2 \leqslant \| t_p(e^2) - x^2 \| + 2c \| t_p(e^1) - x \| + c^2 \| t_p(e^0) - 1 \|,$$

where $c = \max(|a|, |b|)$. If $\| t_p(e^i) - x^i \| \to 0$ as $p \to \infty$, i.e., if $L_n(e^i)(x)$ is almost convergent to x^i, uniformly in $[a, b]$, for $i = 0, 1, 2$, then $\mu_p \to 0$ and we obtain from Theorem 1 that $\{L_n(f)\}$ is almost convergent to f, uniformly in $[a, b]$.

In $[0, 1]$ the Bernstein polynomial $B_n(f)(x)$ converges to $f(x) \in C[0, 1]$ uniformly and a fortiori is almost convergent to f, uniformly there. Since $B_n(e^0)(x) = 1$, $B_n(e^1)(x) = x$, and $B_n(e^2)(x) = (n - 1)\, x^2/n$, we have.

$$B_n(t - x)^2\, (x) = (x - x^2)/n.$$

Thus

$$\begin{aligned} \mu_p^2 &= \| t_p(t - x)^2\, (x) \| = \sup_k \sup_{x \in [0,1]} |x - x^2| \left|(1/p) \sum_{n=k+1}^{k+p} (1/n)\right| \\ &\sim (\tfrac{1}{4} p) \sup_k \ln(1 + (p/k)) \qquad (k = 1, 2, \ldots) \\ &= \{\ln(1 + p)\}/4p \qquad (p = 1, 2, \ldots). \end{aligned}$$

Hence $\| f - t_p(f) \| \leqslant \bar{K} w(((1 + p)/p)^{1/2})$ $(p = 1, 2, \ldots)$ where $t_p(f)$ is as before with $L_n(f)$ replaced by $B_n(f)$ and $\bar{K}$ is an absolute positive constant not necessarily the same at each occurrence.

Let us denote by $C(K_n)$ the set of all continuous real-valued functions on K_n, a compact subset of R^n. One can modify the proof of Theorem 3 of King and Swetits [7] in a manner similar to that of Volokov [15] in the case of convergence of a sequence of positive linear operators on $C(K_n)$. It can be established that the following $(n + 2)$ functions are test functions for $C(K_n)$ with regard to almost convergence

$$\begin{aligned} f_{0n}(x_1, \ldots, x_n) &= 1; \\ f_{jn}(x_1, \ldots, x_n) &= x_j, \qquad j = 1, 2, \ldots, n; \\ f_{n+1\,n}(x_1, \ldots, x_n) &= x_1^2 + \cdots + x_n^2. \end{aligned}$$

Let $w(\delta)$ denote the modulus of continuity of $f(x_1, \ldots, x_n) \in C(K_n)$ where K_n is a compact and convex subset of R^n. We have

$$w(\delta) = \max_{\substack{x,y \in K_n \\ d(x,y) \leq \delta}} |f(x) - f(y)|,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

By a reasoning parallel to that used in the proof of Theorem 1 we can obtain the following.

THEOREM 2. *Let K_n be a compact and convex subset of R^n and let $\{L_n\}$ be a sequence of positive linear operators on $C(K_n)$. Suppose that $t_p^k(e^0)$ is uniformly bounded in K_n. Let $w(\delta)$ be the modulus of continuity of $f \in C(K_n)$. Then, for $p = 1, 2, \ldots$, we have*

$$\|f - t_p(f)\| \leq \|f\| \cdot \|t_p(e^0) - 1\| + w(\mu_p) \|t_p(e^0) + 1\|$$

where

$$\mu_p = \left\| t_p \left(\sum_{i=1}^{n} (\xi_i - x_i)^2 ; x_1, x_2, \ldots, x_n \right) \right\|^{1/2}. \tag{2.4}$$

If, in particular, $t_p^k(e^0)(x) = 1$, then

$$\|f - t_p(f)\| \leq 2\, w(\mu_p).$$

In (2.4) it should be understood that L_n operates on a function of $\xi_1, \ldots, \xi_n$ and the resulting function is evaluated at a point $(x_1, \ldots, x_n)$ and that $t_p(f)$ is then formed according to (1.1).

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Let $\{f_0, f_1, f_2\}$ be a Chebyshev system on $[a, b]$, i.e., no nontrivial linear combination $\sum_{i=0}^{2} a_i(x) f_i(x)$ has more than two zeros in $[a, b]$, multiplicities being counted. Suppose that for all $x, t \in [a, b]$,

$$\begin{aligned} F(x, t) &= \sum_{k=0}^{2} a_k(x) f_k(t) \geq \overline{K}(t - x)^2; \\ F(x, x) &= 0; \end{aligned} \tag{3.1}$$

where the $a_k(x)$ are bounded real functions on $[a, b]$. In our next result we estimate the degree of almost convergence to f of a sequence of positive linear operators $\{L_n(f)\}$ using a Chebyshev system; interalia we establish the fact that $\{e^0, e^1, e^2\}$ is not the only set of test functions in $C[a, b]$.

THEOREM 3. *Let $\{L_n\}$ be a sequence of positive linear operators on $C[a, b]$. Let $\{f_i\}_{i=0}^2$ be a Chebyshev system on $[a, b]$ such that* (3.1) *holds. Let w be the modulus of continuity of $f \in C[a, b]$. Then*

$$\|f - t_p(f)\| \leqslant \|f\| \cdot \| t_p(e^0) - 1 \| + w(\mu_p)\| t_p(e^0) + 1 \|,$$

where

$$\mu_p = \{\| t_p(F(t, x))\|/\bar{K}\}^{1/2}.$$

If, in particular, $t_p^k(e^0)(x) = 1$, $\|f - t_p(f)\| \leqslant 2w(\mu_p)$.

This can be proved by modifying the proof of Theorem 2 of Shisha and Mond [10] as we have done for Theorem 1.

Remark. If $L_n(f_i)$ is almost convergent to f_i uniformly in $[a, b]$, then by using a reasoning similar to that of the Remark at the end of Section 1, it can be concluded that $\mu_p \to 0$ as $p \to \infty$. Hence by Theorem 3, $L_n(f)$ is almost convergent to f, uniformly in $[a, b]$. This extends Theorem 3 of King and Swetits [7].

4

Let $C^k[a, b]$ be the linear space of real functions on $[a, b]$ whose kth derivative is continuous. This section is concerned with estimates for $\|f - t_p(f)\|$ when $f \in C^1[a, b]$ or $f \in C^2[a, b]$. Our results are given in

THEOREM 4. *Let $\{L_n\}$ be a sequence of positive linear operators on $C[a, b]$. Suppose that $\{t_p^k(e^0)(x)\}$ is uniformly bounded for $x \in [a, b]$, $p = 1, 2,\ldots$; $k = 1, 2,\ldots$. Let $f \in C^1[a, b]$ and set*

$$w_1(\delta) = \sup_{|x-y| \leqslant \delta} |f'(x) - f'(y)|.$$

Then, setting for $p = 1, 2,\ldots$, $\mu_p = \| t_p((t - x)^2)(x)\|^{1/2}$, we have

$$\| t_p(f) - f\| \leqslant \|f\| \cdot \| t_p(e^0) - 1 \| + \|f'\| \cdot \| t_p(e^0)\| \mu_p + w_1(\mu_p)(1 + \| t_p(e^0)\|).$$

If in particular $t_p^k(e^0)(x) = 1$, (4.1) *reduces to*

$$\| t_p(f) - f\| \leqslant (\|f'\| + 2w_1(\mu_p))\, \mu_p. \tag{4.2}$$

If, in addition, $t_p^k(e^1)(x) = x$ *for all* $x \in [a, b]$, *then*

$$\| t_p(f) - f \| \leqslant 2\mu_p w_1(\mu_p). \tag{4.3}$$

THEOREM 5. *Let the hypotheses of Theorem* 4 *hold and, in addition, suppose* $f \in C^2[a, b]$; *then, for* $p = 1, 2, \ldots,$

$$\| t_p(f) - f \| \leqslant \| f \| \| t_p(e^0) - 1 \| + \mu_p \| t_p(e^0) + 1 \| (\| f' \| + \mu_p \| f'' \|).$$

If, in addition, $t_p^k(e^0)(x) = 1$ *and* $t_p^k(e^1)(x) = x$, *then*

$$\| t_p(f) - f \| \leqslant 2\mu_p^2 \| f'' \| .$$

Remark. Theorem 5 follows from Theorem 4 by using the fact that if $f \in C^2[a, b]$ then $w_1(\delta) \leqslant \| f'' \| \delta$.

Proof of Theorem 4. Since $f \in C^1[a, b]$, by the mean value theorem, for any $x, t \in [a, b]$ there exists a β between t and x such that

$$f(t) - f(x) = (t - x) f'(x) + (t - x)\{f'(\beta) - f'(x)\}. \tag{4.4}$$

We have after some familiar simplifications

$$\begin{aligned} | t_p^k(f)(x) &- f(x)\, t_p^k(e^0)(x) | \\ &\leqslant | f'(x) | \cdot | t_p^k(| t - x |)(x) | \\ &\quad + w_1(\delta)\{| t_p^k(| t - x |)(x) | + (1/\delta) | t_p^k(| t - x |^2)(x) |\}. \end{aligned} \tag{4.5}$$

Since for $f, g \in C^1[a, b]$,

$$L_n(fg)(x) \leqslant \{L_n(f^2)(x) \cdot L_n(g^2)(x)\}^{1/2},$$

we have

$$t_p^k(fg)(x) \leqslant \{t_p^k(f^2)(x) \cdot t_p^k(g^2)(x)\}^{1/2}.$$

Using this in (4.5) and proceeding in a manner similar to that in the proof of Theorem 1, (4.1) can be established.

Equation (4.2) follows immediately from (4.1), while (4.3) can be seen from the details of the proof.

Application. Let us apply Theorem 4 to obtain an estimate for the converge of the Szasz–Mirákian operator $P_n(f)$ defined for each $f \in C[0, \infty)$ by

$$P_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} ((nx)^k/k!)\, f(k/n) \qquad (x \in [0, \infty)).$$

Consider $P_n(f)(x)$ for $x \in [0, \lambda], 0 < \lambda < \infty$ and $f \in C^1[0, \lambda]$. Since $P_n(e^0)(x) = 1$ and $P_n(e^1)(x) = x$, we have $\|f - t_p(f)\| \leqslant 2\mu_p\, w_1(\mu_p)$ where $\mu_p = \| t_p(t - x)^2 (x)\|$, the L_n's being replaced by the P_n's in (1.1). Further, on expansion of $P_n(t - x)^2 (x) = x/n$ we get

$$\begin{aligned}\mu_p^2 &= \sup_k \sup_{x \in [0, \lambda]} |x| \left|(1/p) \sum_{n=k+1}^{k+p} (1/n)\right| \\ &\simeq (\lambda/p) \ln(1 + p).\end{aligned}$$

Thus

$$\|f - t_p(f)\| \leqslant \overline{K}((\ln(1 + p)/p))^{1/2}\, w_1(((\ln(1 + p)/p))^{1/2}).$$

Remark. It is worth pointing out that concerning convergence, Stancu [12] has shown that if $f \in C^1[0, \lambda]$, then

$$\sup_{x \in [0, \lambda]} |f(x) - P_n(f)(x)| \leqslant (\lambda + \sqrt{\lambda})\, n^{-1/2} w(n^{-1/2}).$$

5

King and Swetits [7] proved the following

THEOREM A. *Let $\{L_n\}$ be a sequence of positive linear operators on $C_{2\pi}$. The sequence $\{L_n(f)(x)\}$ is almost convergent to $f(x)$, uniformly in* $[0, 2\pi]$, *for each $f \in C_{2\pi}$ if and only if $\{L_n(e^0)(x)\}$, $\{L_n(\cos t)(x)\}$, $\{L_n(\sin t)(x)\}$ are almost convergent respectively to* 1, $\cos x$, *and* $\sin x$, *uniformly in* $[0, 2\pi]$.

It is natural to ask about the degree of almost convergence of $\{L_n(f)(x)\}$ to $f(x)$, uniformly in $[0, 2\pi]$. Our next result answers this and is an analog of a theorem of Shisha and Mond [11] for almost convergence.

Let K be the additive abelian group of real numbers modulo 2π on which we define the metric d_1 in the manner

$$d_1(x, y) = \min\{|x - y|, 2\pi - |x - y|\},$$

for $x, y \in K\, (0 \leqslant x, y \leqslant 2\pi)$. $C(K)$ denotes the set of all continuous real-valued functions defined on K. One can identify a function $f \in C(K)$ with the corresponding continuous 2π periodic functions on $(-\infty, \infty)$. For $f \in C(K)$, we define a modulus of continuity (see Censor [1])

$$w(\delta) = \sup_{\substack{x, y \in K \\ d_1(x, y) \leqslant \delta}} |f(x) - f(y)|. \tag{5.0}$$

We prove

THEOREM 6. *Let $\{L_n\}$ be a sequence of positive linear operators on $C(K)$ with $\{t_p{}^k(e^0)(x)\}$ uniformly bounded in K. With* (5.0), (2.1) *holds where*

$$\mu_p{}^2 = \| t_p(\sin^2((t - x)/2)) \| . \tag{5.1}$$

If, in particular, $t_p{}^k(e^0)(x) = 1$, we get (2.2).

The proof of this theorem can be carried out by adapting the method of Shisha and Mond [11] and Censor [1] with modification necessary for almost convergence as in the proof of our Theorem 1.

Remarks. (1) In forming $\{t_p(\sin^2(t - x)/2)\}$ in (5.1) t is the variable and x is held fixed for each member of the sequence.

(2) Since for $p = 1, 2, \ldots,$

$$\begin{aligned} \mu_p{}^2 \leq (\pi^2/2) \{ \| t_p(e^0) - 1 \| + \{\sup_{x \in K} | \cos x |\} \\ \times \| \cos x - t_p(\cos t)(x) \| + \{\sup_{x \in K} | \sin x |\} \\ \times \| \sin x - t_p(\sin t)(x) \| \}, \end{aligned}$$

the almost convergence of $L_n(f)(x)$ to $f(x)$ when $f = e^0$, cos and sin implies that $\mu_p \to 0$ as $p \to \infty$. Thus $L_n(f)(x)$ is almost convergent to $f(x)$, uniformly in K. This, in effect, yields a proof of Theorem A.

(3) Let $V = K^m$, an m-dimensional torus, be the cartesian product of m unit circles K. Let $C(K^m)$ be the set of all continuous real functions $f(x_1, x_2, \ldots, x_m)$ which are periodic with period 2π in each x_k, $k = 1, 2, \ldots, m$. We define the modulus of continuity for $f \in C(K^m)$ by

$$w(\delta) = \max_{\substack{x, y \in K^m \\ d_m(x, y) \leq \delta}} | f(x) - f(y) |$$

where

$$d_m(x, y) = \left\{ \sum_{i=1}^{m} (d_1(x_i, y_i))^2 \right\}^{1/2},$$

$x = \{x_1, \ldots, x_m\}$ and $y = \{y_1, \ldots, y_m\}$.

It is not difficult to generalize Theorem 6 to m-dimensions by modifying the proof of Theorem 2 of Censor [1]. Such a result would lead to the fact that the $m + 2$ test functions of Morozov [9] for convergence also serve as test functions for almost convergence.

6

Let ϕ be a real continuous periodic function defined on $(-\infty, \infty)$. For $n = 1, 2,\ldots$, let $h_1^n, h_2^n,\ldots, h_n^n$ be given real numbers. Consider the operators σ_n over $C(K)$ defined by

$$\sigma_n(\phi)(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{n} h_k^n(a_k \cos kx + b_k \sin kx)$$

where $\phi(x)$ generates the Fourier series

$$(a_0/2) + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Suppose that for each real x and for $n = 1, 2,\ldots,$

$$\tfrac{1}{2} + \sum_{k=1}^{n} h_k^n \cos kx \geq 0. \tag{6.2}$$

After simplification, we have, for each $\phi \in C(K)$,

$$\sigma_n(\phi)(x) = (1/\pi) \int_{-\pi}^{\pi} \phi(t) \left\{\tfrac{1}{2} + \sum_{k=1}^{n} h_k^n \cos k(t - x)\right\} dt. \tag{6.3}$$

Thus each σ_n is a positive linear operator with $\sigma_n(e^0)(x) = 1$ and consequently $t_p^k(e^0)(x) = 1$ for $k = 1, 2,\ldots,$ and $p = 1, 2,\ldots.$ Also, as

$$\sigma_n(\sin^2((t - x)/2))(x) = (1 - h_1^n)/2,$$

we have

$$t_p^k(\sin^2((t - x)/2))(x) = \tfrac{1}{2}\left(1 - (1/p) \sum_{n=k+1}^{k+p} h_1^n\right)$$
$$= \tfrac{1}{2}(1 - m_p^k),$$

where

$$m_p^k = (1/p) \sum_{n=k+1}^{k+p} h_1^n.$$

As an application of Theorem 6 we have the following

THEOREM 7. *Let $\phi \in C(K)$ have modulus of continuity w. Then*

$$\| \phi - t_p(\phi)\| \leq 2w(\pi(\sup_k | 1 - m_p^k |)/2),$$

where

$$t_p^k(\phi) = (1/p) \sum_{n=k+1}^{k+p} \sigma_n(\phi).$$

Remarks. (1) In view of Theorem 7, the operator $\sigma_n(\phi)$ is almost convergent to ϕ, uniformly in K, if $\sup_k |1 - m_p^k|$ converges to 0 as $p \to \infty$, i.e., if the sequence $\{h_1^n\}$ is almost convergent to 1.

(2) It is interesting to note that if $\{h_1^n\} = \{2, 0, 2, 0, \ldots\}$, then $\{h_1^n\}$, although not convergent, is almost convergent to 1. Hence, there is a sequence of positive linear operators which is not convergent but is almost convergent.

7

Walk [13] has considered the problem of approximating functions which are continuous on some finite interval and have some growth conditions near infinity. Hsu [4] has considered the approximation of unbounded functions on the real line. Wood [14] showed that the results of Hsu on convergence can be extended to almost convergence. Eisenberg and Wood [2] have studied the order of approximation of unbounded functions by positive linear operators and have extended the results of Shisha and Mond [10], and Hsu [4]. We remark that it is possible to give analogs of the results of Eisenberg and Wood [2] for almost convergence. In this connection we prove the following theorem.

THEOREM 8. *Let $\{L_n\}$ be a sequence of positive linear operators on $(-\infty, \infty)$ having a common domain D. Let $e^2 \in D$ $(i = 0, 1, 2)$ and $f \in C(-\infty, \infty) \cap D$. Let $L_n(e^0)(x) = 1$. Let there exist a number $p > 1$ and a positive increasing function Ω such that $\Omega^p \in D$ and $f(t) = 0(\Omega(t))$ $(|t| \to \infty)$. Then on (a, b),*

$$|t_p^k(f)(x) - f(x)| \leqslant 2w(\mu_p) + \bar{K}\,|t_p^k(\Omega^p)(x)|^{1/p}$$

$$\times (\mu_p)^{2/p'} m_x^{-2/p'} + |f(x)|\, m_x^{-2}\mu_p^2$$

where

$$\mu_p^2 = |t_p((t - x)^2)(x)|,$$

$$m_x = \min(|a - x|, |b - x|),$$

$$p' = p/(p - 1).$$

Proof. As in Eisenberg and Wood [2] we have, for $x \in [a, b]$ and $t \in (-\infty, \infty)$, $\delta > 0$,

$$|f(t) - f(x)| \leqslant \left(1 + \frac{|t - x|^2}{\delta^2}\right) w(\delta) + \frac{(t - x)^2}{m_x^2}\{\bar{K}\Omega(t) + |f(x)|\}.$$

Since

$$L_n(fg)(x) \leqslant (L_n(f^p)(x))^{1/p}\,(L_n(g^{p'})(x))^{1/p'},$$

we have by Hölder's inequality,

$$(1/p) \sum_{n=k+1}^{k+p} L_n(fg) \leqslant \{(t_p{}^k(f^p)(x))^{1/p} (t_p{}^k(g^{p'})(x))^{1/p'}\}$$

and consequently

$$t_p{}^k(fg)(x) \leqslant (t_p{}^k(f^p)(x))^{1/p} (t_p{}^k(g^{p'})(x))^{1/p'}.$$

Hence after some simplification

$$|t_p{}^k(f)(x) - f(x)| \leqslant w(\delta)\{1 + (\mu_p/\delta)^2\} + \bar{K}\, |t_p{}^k(\Omega^p)(x)|^{1/p} \times \mu_p^{2/p'} m_x^{-2/p'} + |f(x)|\, m_x^{-2} \mu_p^{2}.$$

If $\mu_p \neq 0$ choose $\delta = \mu_p$ and the result follows. If $\mu_p = 0$ modify the proof as in that of Theorem 1.

Remark. If a sequence of linear operators is not positive on $(-\infty, \infty)$ or on $[0, \infty)$ but is on $[0, b]$ $(0 < b < \infty)$, then the results on degree of approximation true on a finite interval can be extended to the cases $(-\infty, \infty)$ and $[0, \infty)$ by a multiplier enlargement method (see [3, 4, 5]).

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